Surface Waves on a Rotating Fluid

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Small-amplitude waves on the free surface of a liquid rotating at constant angular velocity about a vertical axis are studied analytically and experimentally In the experiments, surface waves are produced by mounting a rotating cylinder partially filled with water on a vibration table and oscillating the entire assembly vertically Subharmonic waves are found to occur as in the nonrotating case, but in the presence of rotation a larger forcing amplitude is required to produce instability A synchronous response, i e , at the forcing frequency, not present in the nonrotating case is also predicted and observed Depending on whether the surface wave frequency is greater or less than twice the rotational speed, the governing field equation is either elliptic or hyperbolic The elliptic case is similar to the nonrotating case, and the disturbance decays rapidly with depth In the hyperbolic case there is no decay of the motion with depth The results may be of interest in connection with sloshing in missile fuel tanks

I Introduction

WAVES on the surface of a rotating fluid have been of interest in meteorological and tidal problems for some time and more recently in connection with sloshing in missile fuel tanks The case of a tidal basin under uniform rotation about a vertical axis was considered by Kelvin[1] and Lamb[2] on the assumptions of shallow-water theory In a fuel tank, shallow-water theory is not applicable, and Miles[3] extended the analysis to a fluid of arbitrary depth for this purpose However, Miles retained the assumption of small slope of the free surface, implying a relatively slow rate of rotation The present analysis removes this restriction and considers, in addition, the excitation and stability of the various modes under a forced vertical vibration of the tank Experimental results confirm the theoretical predictions for both the natural frequencies and stability of various modes On the whole, theoretical and experimental frequencies are in close agreement with Miles' theory [3]

The waves considered are small oscillations of the free surface, which is parabolic in the presence of a uniform rotation and gravity In the absence of gravity, the free surface becomes a cylindrical cavity, and free vibrations in this case have been studied by Phillips[4] and by Miles and Troesch [5] Forced vibrations of rotating fluids with no free surface present have been considered by a number of authors [6–10] In such motions and in the surface-wave case, the governing field equation is either elliptic or hyperbolic, depending on whether the frequency of the vibratory motion is greater or less than twice the rotational frequency For surface waves, the elliptic case is similar to the nonrotating case, and the disturbance decays rapidly with depth In the hyperbolic case, the oscillations do not die out away from the free surface

The results may be of interest in situations where the gravity field is small and a rotation is applied to the tank The results probably do not have much bearing on the vibrations of the vortex formed during draining of a tank, since the rotational motion is then quite different from the rigid-body rotation assumed

II Formulation

A circular cylindrical tank partially filled with an inviscid incompressible liquid is considered rotating about its vertical axis with angular velocity $\boldsymbol{\omega}$ A forced vertical harmonic motion is assumed to be superimposed with an acceleration $Ng \sin\Omega t$, where g is the acceleration of gravity, N is a positive number, Ω is the forcing frequency The motion of the fluid is described by a velocity field $\mathbf{q}$ relative to the tank plus the rigid-body motion of the tank The velocity $\mathbf{q}$ thus defined is relative to a coordinate system moving vertically with the forced harmonic motion and rotating with angular velocity ω

The equation of motion in the moving coordinate system is

$$(\partial \mathbf{q}/\partial t) + (\mathbf{q} \cdot \nabla)\mathbf{q} + 2\boldsymbol{\omega} \times \mathbf{q} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) = -\nabla[(p/\rho) + gz(1 + N \sin\Omega t)] \quad (2\ 1)$$

where $\mathbf{R}$ is the position vector, p the pressure, ρ the mass density, and z the vertical coordinate measured upwards It is convenient to work in terms of an acceleration potential χ defined by

$$\chi = (p/\rho) - gz_0 + gz(1 + N \sin\Omega t) \quad (2\ 2)$$

where z_0 is given by

$$z_0 = (\omega^2/2g)(r^2 - \tfrac{1}{2}a^2) \quad (2\ 3)$$

The scalar r is the distance from the axis of rotation, and a is the radius of the cylinder The plane free surface for the system at rest is chosen as $z = 0$, so that $z = z_0$ is the free surface under rigid-body rotation

The convective term $(\mathbf{q} \cdot \nabla)\mathbf{q}$ in (2 1) is neglected on the grounds that the wave motion is small; then, on the assumption of a harmonic time factor $e^{i\sigma t}$ in $\mathbf{q}$ and χ, Eq (2 1) may be written as

$$i\sigma\mathbf{q} + 2\boldsymbol{\omega} \times \mathbf{q} = -\nabla\chi \quad (2\ 4)$$

Equation (2 4) can be solved for $\mathbf{q}$ in terms of χ Let u, v, w be the components of $\mathbf{q}$ with respect to cylindrical coordinates r, θ, z Then

$$u = -i(\sigma)^{-1}(1 - \mu^2)^{-1}\left(\frac{\partial\chi}{\partial r} - \frac{i\mu}{r}\frac{\partial\chi}{\partial\theta}\right)$$

$$v = -(i\sigma)^{-1}(1 - \mu^2)^{-1}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta} + i\mu\frac{\partial\chi}{\partial r}\right) \quad (2\ 5)$$

$$w = -(i\sigma)^{-1}\frac{\partial\chi}{\partial r}$$

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where

$$\mu = 2\omega/\sigma \tag{2 6}$$

and σ is the frequency of the surface waves

Substituting (2 5) in the equation o continuity, $\nabla \cdot \mathbf{q} = 0$, yields the final field equation on χ:

$$\nabla^2\chi - \mu^2(\partial^2\chi/\partial z^2) = 0 \tag{2 7}$$

Equation (2 7) with $N = 0$ reduces to the case treated in Ref 3

Solutions of (2 7) are required satisfying appropriate boundary conditions On the free surface the pressure is zero However, to include the effect of damping in an approximate manner, it is convenient to postulate a pressure at the surface proportional to the vertical velocity of the surface and always opposing the motion Since the slope of the free surface is ω^2r/g, the velocity of the free surface normal to itself is

$$[w - u(\omega^2r/g)][1 + (\omega^2r/g)^2]^{-1/2} \tag{2 8}$$

The vertical velocity of the surface is then

$$w - u(\omega^2r/g) \tag{2 9}$$

Replacing p in (2 2) by $\rho\nu$ times (2 9), where ν is a pseudoviscosity, the dynamical surface condition is

$$\chi = \nu[w - u(\omega^2r/g)] + Ngz_0 \sin\Omega t + g\zeta(1 + N \sin\Omega t) \tag{2 10}$$

to be satisfied on $z = z_0 + \zeta$, where $\zeta(r, t)$ is the vertical displacement of the surface A kinematic condition must also be satisfied on the free surface, namely, that the suıface velocity $\partial\zeta/\partial t$ be equal to (2 9):

$$\partial\zeta/\partial t = w - u(\omega^2r/g) \text{ on } z = z_0 + \zeta \tag{2 11}$$

Assuming a harmonic time factor $e^{i\sigma t}$, (2 10) and (2 11) can be combined in the form

$$\chi = Ngz_0 \sin\Omega t + [g(1 + N \sin\Omega t)(i\sigma)^{-1} + \nu] \times [w - u(\omega^2r/g)] \tag{2 12}$$

Equation (2 12) will be applied on $z = z_0$ rather than $z = z_0 + \zeta$, since ζ is assumed to be small

The boundary condition on the walls and bottom of the tank is, of course, that the normal component of $\mathbf{q}$ vanishes

III Solution for the Elliptic Case

When μ is less than unity, the equation on χ, (2 7), is elliptic In this case a solution can be achieved by separation of variables using parabolic coordinates ξ_1, ξ_2 defined in terms of r, z by

$$\begin{aligned}\xi_1 &= \{[r^2 + (z + h)^2(1 - \mu^2)^{-1}]^{1/2} + (z + h)(1 - \mu^2)^{-1/2}\}^{1/2} \\ \xi_2 &= \{[r^2 + (z + h)^2(1 - \mu^2)^1]^{1/2} - (z + h)(1 - \mu^2)^{-1/2}\}^{1/2}\end{aligned} \tag{3 1}$$

where the positive square root is taken throughout The constant h locates the parabolic coordinates with respect to the origin of z and will be selected so that the parabolic free surface coincides with a surface $\xi_2 = $ const Figure 1 shows coordinates, ξ_1, ξ_2 for $h = 0$

In parabolic coordinates, Eq (2 7) becomes

$$\frac{\partial}{\partial\xi_1}\left(\frac{\xi_1\xi_2}{\sin\theta}\frac{\partial\chi}{\partial\xi_1}\right) + \frac{\partial}{\partial\xi_2}\left(\frac{\xi_1\xi_2}{\sin\theta}\frac{\partial\chi}{\partial\xi_2}\right) + \frac{\partial}{\partial(\cos\theta)}\frac{(\xi_2^2 + \xi_1^2)\sin\theta}{\xi_1\xi_2}\frac{\partial\chi}{\partial(\cos\theta)} = 0 \tag{3 2}$$

which is separable assuming that $\chi = \chi_1(\xi_1)\chi_2(\xi_2)e^{i(\sigma t + m\theta)}$ where m is an integer The separated equations are

$$\frac{1}{\xi_1}\frac{d}{d\xi_1}\left(\xi_1\frac{d\chi_1}{d\xi_1}\right) + \left(k^2 - \frac{m^2}{\xi_1^2}\right)\chi_1 = 0 \tag{3 3}$$

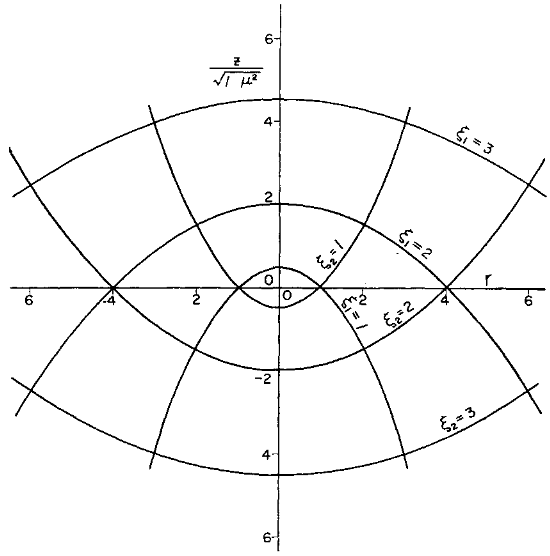

Fig 1 Parabolic cooıdinates, elliptic case

$$\frac{1}{\xi_2}\frac{d}{d\xi_2}\left(\xi_2\frac{d\chi_2}{d\xi_2}\right) - \left(k^2 - \frac{m^2}{\xi_2^2}\right)\chi_1 = 0 \tag{3 4}$$

where k is a separation constant The general solution of (3 3) is a linear combination of the Bessel functions $J_m(k\xi_1)$ and $Y_m(k\xi_1)$, but Y_m is not used here because it is singular at $\xi_1 = 0$ A linear combination of modified Bessel functions $I_m(k\xi_2)$ and $K_m(k\xi_2)$ satisfies (3 4), but I_m is not used because solutions that decay with depth are anticipated The solution considered is

$$\chi = A_mJ_m(k\xi_1)K_m(k\xi_2)e^{i(\sigma t + m\theta)} \tag{3 5}$$

where A_m is a complex amplitude

Consider the free surface to be given by $\xi_2 = \text{const} = \bar{\xi}_2$ The inverse of (3 1) is

$$\begin{aligned} r &= \xi_1\xi_2 \\ z &= \tfrac{1}{2}(1 - \mu^2)^{1/2}(\xi_1^2 - \xi_2^2) - h\end{aligned} \tag{3 6}$$

Substituting $\bar{\xi}_2$ for ξ_2 and z_0 from (2 3) for z yields an equation that must hold for all ξ_1:

$$(\xi_1^2 - \bar{\xi}_2{}^2)(1 - \mu^2)^{1/2} - 2h = (\xi_1^2\bar{\xi}_2{}^2 - \tfrac{1}{2}a^2)\omega^2/g \tag{3 7}$$

which requires

$$\bar{\xi}_2 = [g(1 - \mu^2)^{1/2}/\omega^2]^{1/2} \tag{3 8}$$

and

$$h = \omega^2a^2/4g - g(1 - \mu^2)/2\omega^2 \tag{3 9}$$

The free surface condition (2 12) now applied on $\xi_2 = \bar{\xi}_2$ may be written in parabolic coordinates using Eqs (2 5, 3 1, 3 8, and 3 9) The result is

$$\chi = [i\sigma\bar{\xi}_2(1 - \mu^2)^{1/2}]^{-1}[g(1 + N \sin\Omega t)(i\sigma)^{-1} + \nu] \times [(\partial\chi/\partial\xi_2) + m\mu\chi/\bar{\xi}_2] + \tfrac{1}{2}N\omega^2(\xi_1^2\bar{\xi}_2{}^2 - \tfrac{1}{2}a^2) \sin \Omega t \text{ on } \xi_2 = \bar{\xi}_2 \tag{3 10}$$

Free Vibrations

Substituting (3 5) into (3 10) with $N = \nu = 0$ yields the frequency equation for free, undamped surface waves:

$$k\bar{\xi}_2 K_m'(k\bar{\xi}_2) + [4(\mu^{-2} - 1) + m\mu]K_m(k\bar{\xi}_2) = 0 \tag{3 11}$$

For $m = 0$, the roots of (3 11) determine the frequency σ of axisymmetric waves as a function of the wave number k Numerical results are plotted in Fig 2 in dimensionless form in terms of $\mu = 2\omega/\sigma$ and λ defined by

$$\lambda = kg\omega^{-1}(1 - \mu^2)^{1/4} \tag{3 12}$$

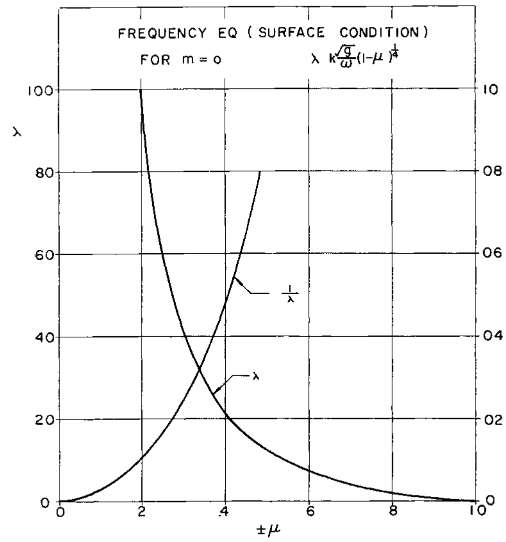

Fig 2 Wave number vs fiequency, $m = 0$, elliptic case

If $m = \pm M$, where M is a positive integer, two different values of σ are found satisfying (3 11) for given values of M and k, depending on whether $m = +M$ or $m = -M$ This indicates that waves propagating in the direction of rotation represented by $e^{i(\sigma t - M\theta)}$ have a different frequency than waves of the same shape traveling in the opposite direction, represented by $e^{i(\sigma t + M\theta)}$ This phenomenon is referred to as frequency splitting[3] and is due to the rotation The roots of (3 11) for $m = \pm 1$ and $m = \pm 2$ are shown in Figs 3 and 4

The waves just described are possible motions of a rotating fluid extending to infinity in r and in depth These waves may also take place in containers that are made up of surfaces ξ_1 = const and ξ_2 = const (Fig 1) If the side wall is given by ξ_1 = const = ξ_{1w}, the condition $q_n = 0$ at the wall is

$$k\xi_{1w}J_m{}'(k\xi_{1w}) + m\mu J_m(k\xi_{1w}) = 0 \qquad (3\ 13)$$

Equation (3 13) determines the wave number k once μ, m, and ξ_{1w} are given Solutions of (3 13) are plotted in Figs 5 and 6 for $m = \pm 1$ and $m = \pm 2$ These figures show the first two of the infinite number of branches which exist for each m These solutions do not satisfy the boundary conditions for a circular cylindrical tank with vertical walls However, if the radius at the intersection of $\xi_1 = \xi_{1w}$ and the free surface is taken equal to the cylinder radius a, the theoretical frequencies agree very well with experimental values for a circular cylindrical tank

The boundary condition at the bottom of the tank is neglected because (3 5) decays rapidly with depth The bottom boundary condition could be satisfied by adding a term involving $I_m(k\xi_2)$ to (3 5), but the correction will be small except for shallow depths of fluid

Forced Vibrations

With $N > 0$, the free surface condition (3 10) determines the response of the fluid to a vertical vibration of the tank Forced vibrations are considered by setting the surface wave frequency σ equal to the forcing frequency Ω The solution may be written as a Fourier-Bessel series of the form

$$\chi = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} A_{mn}J_m(k_{mn}\xi_1)K_m(k_{mn}\xi_2)e^{i(\Omega t + m\theta)} \qquad (3\ 14)$$

where the k_{mn} are the roots of (3 13) Equation (3 14) is to be substituted into the free surface condition (3 10) Before doing so, the term $N \sin\Omega t$ in the factor $(1 + N \sin \Omega t)$ is dropped from (3 10) because the resulting disturbance that has been assumed small is proportional to N In the last term of (3 10), the factor $N \sin\Omega t$ is retained as it is the forcing

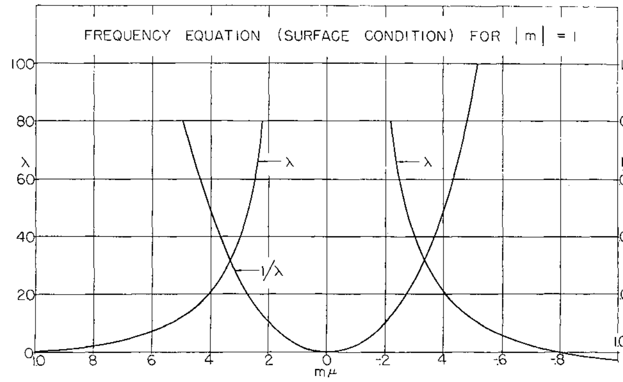

Fig 3 Wave number vs frequency, $m = \pm 1$, elliptic case

function, but it is replaced by $Ne^{i\Omega t}$ The forcing term is also expanded into a Fourier-Bessel series using

$$\left(\xi_1{}^2\xi_2{}^2 - \frac{1}{2}a^2\right) = 4\,\xi_2{}^2 \sum_{n=1}^{\infty} \frac{J_0(k_{0n}\xi_1)}{k_{0n}{}^2J_0(k_{0n}\xi_{1w})} \qquad (3\ 15)$$

Substituting (3 14) and (3 15) into (3 10) and equating coefficients of $J_m(k_{mn}\xi_1)e^{im\theta}$ allows solution for the amplitudes A_{mn} Since the forcing term (3 15) contains only $m = 0$, it follows that $A_{mn} = 0$ for $m \neq 0$ Hence the motion is axisymmetric For $m = 0$,

$$A_{0n} = 2N\omega^2\xi_2{}^2[k_{0n}{}^2J_0(k_{0n}\xi_{1w})]^{-1}\Big[K_0(k_{0n}\xi_2) + \frac{(g/i\sigma) + \nu}{i\sigma(1 - \mu^2)^{1/2}\xi_2}\,k_{0n}K_1(k_{0n}\xi_2)\Big]^{-1} \qquad (3\ 16)$$

It is notable that every axisymmetric mode shape is excited at every forcing frequency Resonance occurs at each frequency of free vibration, but the amplitudes are finite for $\nu > 0$ The motion goes to zero as ω approaches zero and has no counterpart in a nonrotating fluid

Subharmonics

The response of a nonrotating fluid to vertical vibration is zero except for narrow ranges of the forcing frequency centered at σ, 2σ, 4σ, , etc , where σ is a frequency of a mode of free vibration [11] Within these zones, the plane free surface is unstable, and the response is infinite according to linear theory, but is in fact limited by nonlinear effects [12] The $\frac{1}{2}$ subharmonic, i e , the response at forcing frequency 2σ, is the most prominent and is the only response easily produced experimentally [12]

The $\frac{1}{2}$ subharmonic is also predicted and observed for a rotating fluid To study the $\frac{1}{2}$ subharmonic, σ is set equal to $\frac{1}{2}\Omega$ In this case, the term $(1 + N \sin\Omega t)$ in (3 10) is retained, since the existence of the $\frac{1}{2}$ subharmonic depends on the parametric excitation due to this term, and the amplitude of the forced vertical vibration of the tank need not be small It is convenient to study the stability of one mode shape at a time

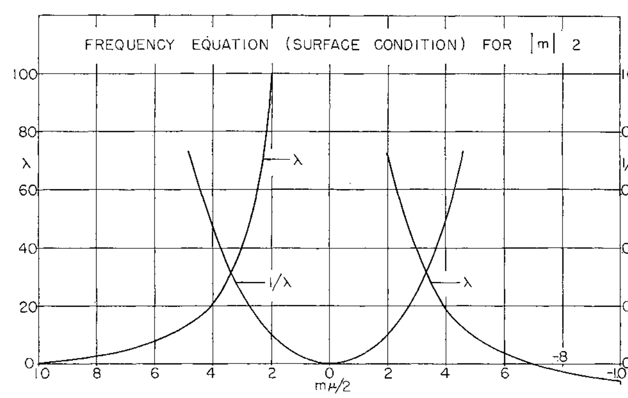

Fig 4 Wave number vs frequency, $m = \pm 2$, elliptic case

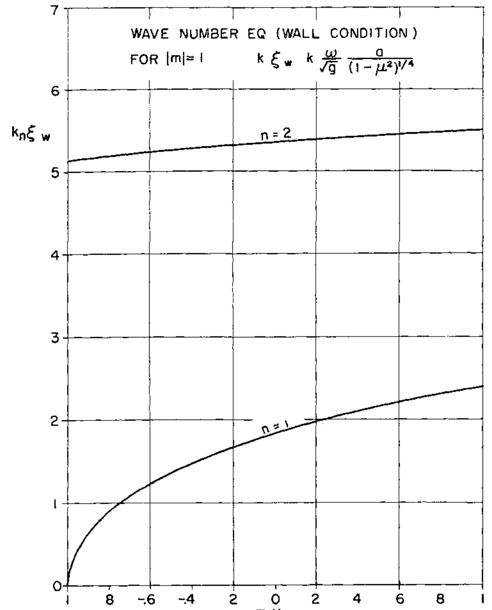

Fig 5 Wave numbers for finite tanks, $m = \pm 1$, elliptic case

by assuming χ to be approximated by

$$\chi = [(A_J + iA_2)e^{i(\sigma t + m\theta)} + (A_3 + iA_4)e^{i(\sigma t \quad m\theta)}] \times [J_m(k\xi_1)K_m(k\xi_2)] \quad (3\ 17)$$

where the A_i are real, m is a positive integer, k is a root of (3 13), and o is to be taken equal to $\frac{1}{2}\,\Omega$ Equation (3 17) is approximate in that a more general periodic wave form would require Fourier series representation of which (3 17) is the first term After substitution of (3 17) into (3 10), the various trigonometric functions appearing are expanded into terms each containing one of the products $\sin\sigma t\ \sin m\theta$, $\sin\sigma t\ \cos m\theta$, $\cos\sigma t\ \sin m\theta$, $\cos\sigma t\ \cos m\theta$, or higher harmonics of σt The higher harmonics are neglected because (3 17) is only the first term of a complete Fourier series This general procedure for the study of subharmonics is discussed by McLachlan [13] Equating coefficients of each of the four trigonometric products

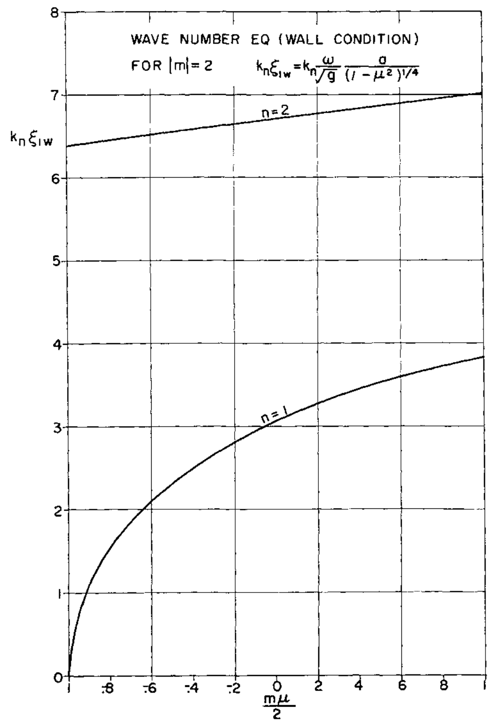

Fig 6 Wave numbers for finite tanks, $m = \pm 2$, elliptic case

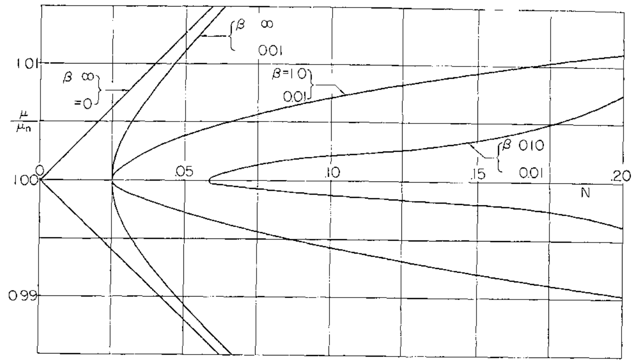

Fig 7 Subharmonic steady states, $m = 0$, elliptic case

previously listed to zero yields four equations on the unknown amplitudes A_i

$$\begin{aligned} &\nu B_1A_1 + (B_3 + gB_1/\sigma)A_2 - (Ng/2\sigma)B_2A_3 = 0 \\ &(B_3 + gB_1/\sigma)A_1 - \nu B_1A_2(Ng/2\sigma)B_2A_4 = 0 \\ &(Ng/2\sigma)B_1A_1 - \nu B_2A_3 - (B_3 + gB_2/\sigma)A_4 = 0 \\ &(Ng/2\sigma)B_1A_2 - (B_3 + gB_2/\sigma)A_3 + \nu B_2A_4 = 0 \end{aligned} \quad (3\ 18)$$

where the constants B_i are

$$B_1 = [kK_m'(k\xi_2) + (m\mu/\xi_2)K_m(k\xi_2)](\sigma\xi_2)^{-1}(1 - \mu^2)^{-1/2}$$

$$B_2 = [kK_m'(k\xi_2) - (m\mu/\xi_2)K_m(k\xi_2)](\sigma\xi_2)^{-1}(1 - u^2)^{-1/2}$$

$$B_3 = K_m(k\xi_2) \quad (3\ 19)$$

The determinant of the coefficients of the A_i in (3 18) must be zero for nontrivial solutions The resultant equation is interpreted as specifying the acceleration N required to maintain steady-state vibrations for a given ω, a, k, ν, and a particular frequency Ω The set of possible steady states traces a curve in the Ω,N plane It is assumed that this curve separates regions for which the parabolic free surface is stable or unstable as in the similar situation of a nonrotating fluid[12] and other applications [13]

For axisymmetric waves, $m = 0$, $B_1 = B_2$, and the value of N obtained from the determinant of (3 18) is

$$N = (2\sigma/gB_1)[(B_3 + gB_1/\sigma)^2 + \nu^2B_1^2]^{1/2} \quad (3\ 20)$$

Numerical values of N computed from (3 20) are shown in Fig 7; the forcing frequency Ω is represented by the ordinate μ/μ_n, where μ_n is the value of μ at the natural frequency corresponding to the given value of k The dimensionless parameters β and ν' for the different curves are defined by

$$\beta = kg^{1/2}/\omega \qquad \nu' = \nu kg^{-1/2} \quad (3\ 21)$$

ν' is a dimensionless damping coefficient, whereas β is the reciprocal of a Froude number The nonrotating case corresponds to $\beta = \infty$, for which the results are known [11] The region between the two branches of any curve represents unstable cases, whereas the remainder of the plane represents stable points for which there is no subharmonic response The effect of increasing speed of rotation corresponding to decreasing β is to narrow the unstable region generally, and in this sense the rotation is a stabilizing influence

Asymmetric modes, $m \neq 0$, describe the so-called sloshing motions of which the lowest and principal mode is $m = 1$ The determinant of (3 18) is more complicated than for $m = 0$, but the general character of the results is the same There are $\frac{1}{2}$ subharmonics possible when the forcing frequency is in the vicinity of twice a natural frequency For the undamped case $\nu = 0$, the steady states are given by

$$N = (2\sigma/g)(B_1B_2)^{-1/2}[B_3 + gB_1/\sigma]^{1/2}[B_3 + gB_2/\sigma]^{1/2} \quad (3\ 22)$$

Numerical results for $m = 1$ are shown in Fig 8 The fre-

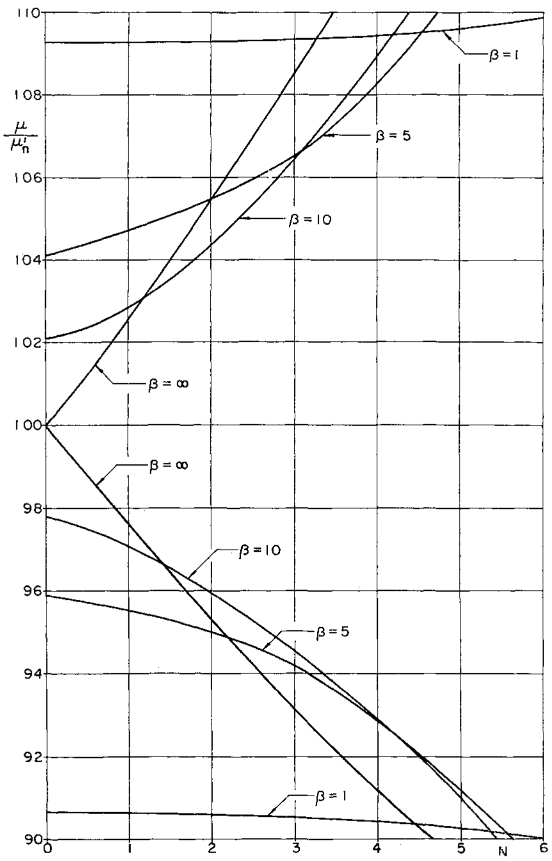

Fig 8 Subharmonic steady states, $m = 1$, elliptic case

quency ordinate (μ/μ_n') has been normalized using the mean frequency μ_n' of the free vibration frequencies of forward and backward moving waves The curves for $\beta = \infty$ again correspond to the nonrotating case For $\beta < \infty$, the intersections of the curves shown with the vertical axis represent the free vibration frequencies In this case, the spread of the free vibration frequencies increases the range of instability at low values of n

IV Solution for the Hyperbolic Case

When μ^2 is greater than unity, the equation on χ, (2 7), is hyperbolic In this case, solutions that do not decay with depth are found As a consequence, the wave forms and frequencies are strongly dependent on container shape, and only limited classes of solutions have been achieved

The characteristic surfaces of (2 7) are the cones

$$z = \pm(\mu^2 - 1)^{1/2}r + c \tag{4 1}$$

where c is a constant A region R_0 is defined as the space between the parabolic free surface and the characteristic cone that is tangent to this surface Solutions of the first category found are limited to the region R_0 For this purpose, define

$$\begin{aligned} \eta_1 &= \{(z + b)(\mu^2 - 1)^{-1/2} - [-r^2 + (z + b)^2(\mu^2 - 1)^{-1}]^{1/2}\}^{1/2} \\ \eta_2 &= \{(z + b)(\mu^2 - 1)^{-1/2} + [-r^2 + (z + b)^2(\mu^2 - 1)^{-1}]^{1/2}\}^{1/2} \end{aligned} \tag{4 2}$$

where the positive square root is taken throughout, and b is a constant The coordinates η_1, η_2 are not orthogonal, and they are real only in the region above the characteristic cone (4 1) with $c = -b$, but it is found that (2 7) is separable in these coordinates Free vibrations, forced vibrations, and subharmonic waves may be studied for containers lying entirely in R_0 with sides and bottoms given by η_1 = const and η_2 = const One form of the solutions found is

$$\chi = C_m J_m(k\eta_2) J_m(k\eta_1) e^{i(\sigma t + m\theta)} \tag{4 3}$$

where C_m is a complex constant The frequency equation for free vibrations and solutions for forced and subharmonic waves are given in Ref 14 but are of limited interest, because it is not found tractable to extend them below the region R_0 In principle, by the use of combinations and extensions of the modes found, it is possible to satisfy boundary conditions for arbitrary containers extending beyond R_0 But, in practice, this has not been achieved in any particular case

A second category of solutions possible in the hyperbolic case is for slow rotations in which the slope of the free surface is small Then the boundary condition (2 12) may be applied on the surface $z = 0$, since z_0 (2 3) is very nearly zero Furthermore, (2 12) may be reduced to

$$\chi = Ngz_0 \sin\Omega t + g(1 + N \sin\Omega t)(i\sigma)^{-1}w \quad \text{on} \quad z = 0 \tag{4 4}$$

for the case $\nu = 0$ The free vibrations possible within this approximation were given by Miles,[3] and the treatment is extended here to include forcing by vertical vibration

Consider a cylinder of radius a and mean fluid depth d The appropriate solution of (2 7) satisfying the condition of zero normal velocity at the bottom $z = -d$ is

$$\chi = D_m J_m(kr) e^{i(\sigma t + m\theta)} \cos[k(z + d)(\mu^2 - 1)^{-1/2}] \tag{4 5}$$

where D_m is a complex constant and k is determined by the condition $u = 0$ at $r = a$ which takes the form

$$kaJ_m'(ka) + m\mu J_m(ka) = 0 \tag{4 6}$$

The frequency equation of free vibrations is found on substitution of (4 5) into (4 4) with $N = 0$:

$$\sigma_n^2 = -k_n g(\mu^2 - 1)^{-1/2} \tan[k_n d(\mu^2 - 1)^{-1/2}] \tag{4 7}$$

Some roots of (4 6) and (4 7) are given in Ref 3

Although the surface is very nearly flat, a response at the forcing frequency is still obtained in the present case when the container is vibrated vertically By expanding z_0 from (2 3) in a Fourier-Bessel series of terms of the form (4 5), it is found that the response must be axisymmetric, and (4 4) yields the amplitude coefficients The real solution achieved is of the form (4 5) with $D_n e^{i\sigma t}$ replaced by $D_{0n} \sin\sigma t$, where D_{0n} is real and given by

$$D_{0n} = 2N\omega^2[k_{0n}{}^2 J_0(k_{0n}a)(1 - \sigma_n{}^2/\sigma^2) \times \cos(k_{0n}d)(\mu^2 - 1)^{-1/2}]^{-1} \tag{4 8}$$

This forced vibration is similar to the elliptic case in that it is proportional to N and contains the usual resonance phenomena

Subharmonic response also may be produced as in the elliptic case In place of (3 17), in the present case it is assumed

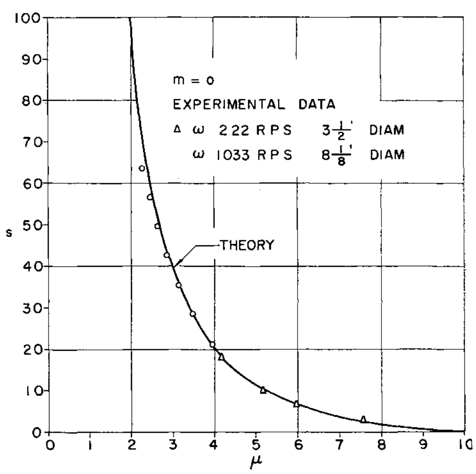

Fig 9 Comparison of theory and experimental natural frequencies, $m = 0$, elliptic case

that

$$\chi = [(D_1 + iD_2)e^{i(\sigma t + m\theta)} + (D_3 + iD_4)e^{i(\sigma t - m\theta)}] \times J_m(kr) \cos[k(z + d)(\mu^2 - 1)^{-1/2}] \quad (4\ 9)$$

where the D_i are real, m is positive, k is a root of (4 6), and σ is to be set equal to $\frac{1}{2}\Omega$ for $\frac{1}{2}$ subharmonics Substituting (4 9) into the free surface condition (4 4) and using the same procedure as in the elliptic case, an equation for the steady states may be derived For the case $m = 0$, the end result is

$$N = \pm 2(1 - \sigma^2/\sigma_n^2) \quad (4\ 10)$$

This is the same result as for the undamped, nonrotating elliptic case shown in Fig 7 with $\beta = \infty$ and $\nu' = 0$ Equation (3 20) reduces to (4 10) when $\nu = 0$ and $\omega = 0$ The steady states (4 10) separate stable and unstable states in the N,σ plane, as usual for such subharmonics

V Experimental Results

Experiments were performed using circular Lucite tanks mounted on bearings on a vibration table so that the tank could be simultaneously rotated about its vertical axis by an electric motor and vibrated vertically Tanks 3 50 and 8 125 in in diameter and 24 in high were used filled to variable depths with water To begin a test, the tank was first rotated at constant speed until the water was rotating as a rigid body Then a vertical vibration was applied The various modes generated were identified by observing the number of nodes of the free surface wave shape Both forced waves at the forcing frequency and $\frac{1}{2}$ subharmonics at one-half the forcing frequency were observed

Resonant frequencies were determined by varying the vibrator frequency until the largest-amplitude wave was observed in a particular mode, at which point the frequency was read on the vibration control cabinet Data for the elliptic case were obtained using a 3 50-in -diam tank rotating at

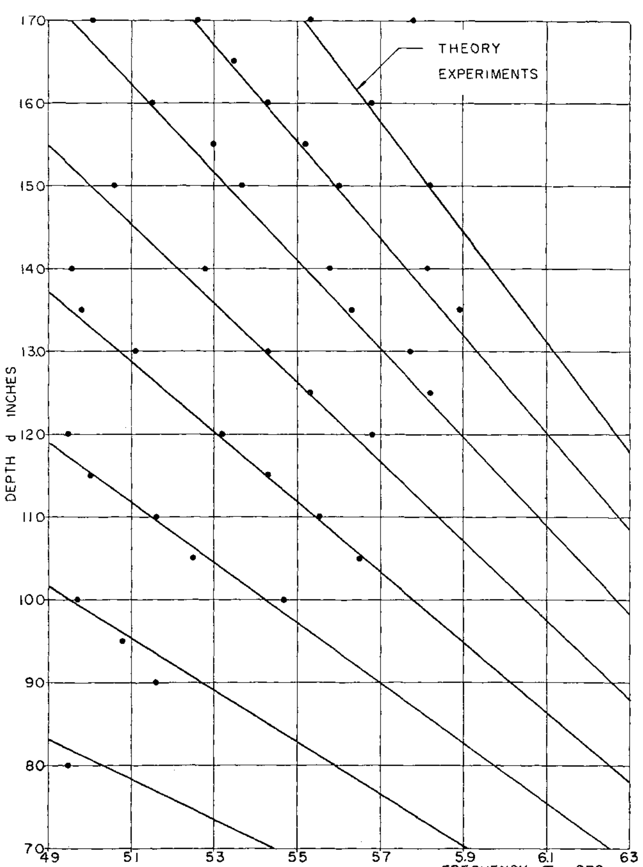

Fig 10 Comparison of theory and experimental natural frequencies, $m = 0$, hyperbolic case

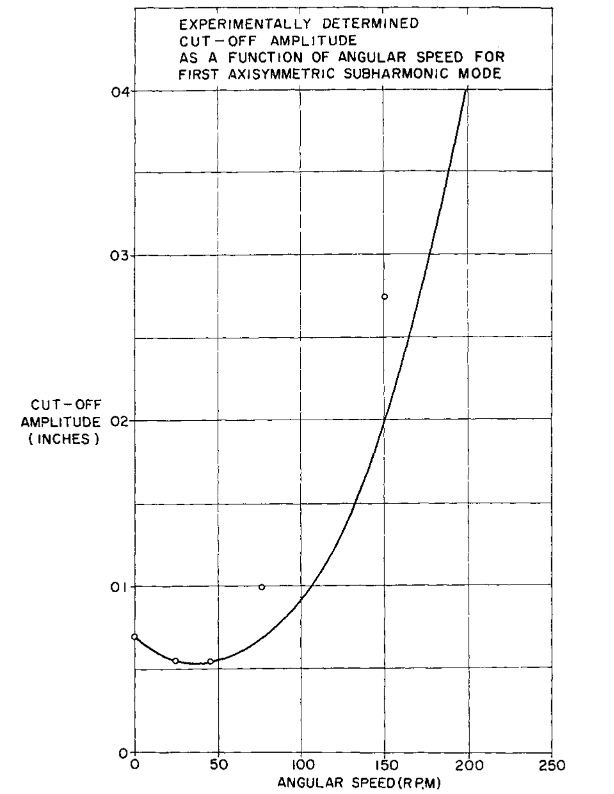

Fig 11 Experimental cut-off amplitude for $\frac{1}{2}$ subharmonic response, $m = 0$, elliptic case

2 22 rps and a 8 125-in diam tank rotating at 1 033 rps The experimental points and the theoretical curve of frequencies of free vibration (3 11), are shown in Fig 9 The different points for a given tank represent different modes, all axisymmetric with depths at least three times the tank radius The agreement of the theory and experimental results is good

For the hyperbolic case, the 3 50-in tank was used rotating at 3 92 rps Several depths of fluid were used, since the depth has a strong influence in this case At each depth, several resonant modes were observed The results are shown in Fig 10 with the theoretical curves based on Eq (4 7) Again, the agreement is good In the hyperbolic case there are many modes close together, and the maxima of amplitudes at resonances were not so clear or pronounced as in the elliptic case

The $\frac{1}{2}$ subharmonics observed were used to test the influence of rotation on the minimum vertical acceleration required to produce instability of the fundamental axisymmetric mode Once a subharmonic was achieved, the vibrator frequency and rotational speed were held fixed while the amplitude of the vertical vibration was gradually reduced until the surface waves disappeared This procedure was repeated at various frequencies until a minimum amplitude of vertical vibration required to support a surface wave was established The cutoff amplitudes so determined are plotted in Fig 11 The results confirm that rotation generally increases the amplitude of the forcing motion required to produce instability

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Mach 8 to 22 Studies of Flow Separations Due to Deflected Control Surfaces

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This paper describes measurements of separation phenomena obtained during a hypersonic aerodynamic test program conducted in the Boeing Company 44-in Hotshot wind tunnel and in the Arnold Engineering Development Center tunnel "B " Test results establish the influence on separation parameters exerted by systematic variations of model geometry and Mach and Reynolds numbers Pressure and heat-transfer measurements are correlated with Schlieren photography, temperature-sensitive paint, and oil-streak flow visualization data Applications of theories are described

Nomenclature

C_f = skin-friction coefficient
C_P = pressure coefficient based on P_0, $(P - P_0)/q_0$
$f_3(1)$ = 1 47, Ref 16
K_3 = correction function to heating rate due to pressure gradient, Ref 10
l_i = interaction length, Ref 16
M = Mach number
n = exponent in expression $P \sim X^n$
P = pressure
S_t = Stanton number based on stagnation enthalpy
T = temperature
u = streamwise velocity
X = streamwise coordinate
y = normal coordinate
δ = turning angle from freestream direction
ξ = correction factor for linear flow, Ref 16

Subscripts

BL = boundary layer
F = flap
FP = undisturbed flat plate
n = normal to spoiler
0 = beginning of interaction
p = plateau
s = separation point
W = wall
∞ = freestream

Presented as Preprint 63-173 at the AIAA Summer Meeting Los Angeles, Calif , June 17–20, 1963; revision received October 23, 1963 The pressure and heat transfer tests being conducted in the Hotshot wind tunnel are part of an internal Boeing aerodynamic research program The Mach 8 results presented were obtained during the course of a joint Air Force Aeronautical Systems Division and Boeing investigation of aerodynamic interference effects Boeing personnel who have contributed significantly to this program are the Boeing wind tunnel design and operations group, E J Sears, and Richard McKenzie The assistance of G Alexander, W Hankey, and J Ondrejka of the Aeronautical Systems Division and the unknown Air Force personnel who expedited the clearance for publication of the Mach 8 data is gratefully acknowledged

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Introduction

FLOW separation effects are not completely understood at low Mach numbers, but the body of empirical knowledge and the theoretical understanding that have been developed can generally prevent, or quickly solve, flight problems resulting from separation This knowledge of separation phenomena from subsonic through low supersonic speeds was acquired progressively over a period of approximately 40 to 50 years The extension of wind tunnel data to flight regimes involved conventional wind tunnel scaling techniques because wind tunnels simulated flight conditions reasonably well

Approximately only 10 years have elapsed from the advent of routine supersonic to orbital flight Thus, in huge steps, flight progress has advanced far beyond present capability to predict confidently the effects of the high Mach numbers